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Abstract

We study normal state properties of an interacting Fermi gas in an isotropic harmonic trap of arbitrary dimensions. We exactly calculate the first-order perturbation terms in the ground-state energy and chemical potential, and obtain simple analytic expressions of the total energy and chemical potential. At zero temperature, we find that the Thomas–Fermi approximation agrees well with the exact results for any dimension even though the system is small. In the high-temperature (classical) region, we find the interaction energy decreases in proportion to $T^{-d/2}$, where T is the temperature and d is the dimension of the system.

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1. Introduction

Recently Fermi degeneracy in neutral atomic gases has been attained for ^{40}K [1] and ^{6}Li [2–4]. In these systems, one can control physical parameters with relative ease. Especially, the interaction strength can be varied in a wide range by using Feshbach resonance [5]. The exciting phenomena of fermion paring are thus expected either in momentum space (Cooper pairing) or in real space (BEC of preformed fermion pairs), leading to weak-coupling or strong-coupling superfluidity [6–9]. This system is also characterized by the existence of a trapping potential, which leads to discrete quantum eigenstates and shell structures. This makes analytic calculations rather difficult. In many practical cases one uses the Thomas–Fermi approximation for analytic calculations.

This method is justified for a system which has a sufficiently large number of particles. Several properties [10, 15–17] such as single-particle excitation and Cooper pair correlation are investigated in this approximation. However it cannot always be applicable for small systems whose level discreteness plays important roles.

The purpose of this paper is to investigate the normal ground-state properties of a dilute interacting Fermi gas in a trap with the use of the exact eigenstate of the harmonic trap

without resorting to the Thomas–Fermi approximation. We assume contact-type inter-particle interaction, and give a systematic method to count up the quantum number of the harmonic potential. A simple analytic expression for the total ground-state energy and the chemical potential is obtained within the first-order perturbation theory.

2. General formalism

We consider a two-component Fermi gas with a short-range interaction in an isotropic harmonic potential. The two components (hyperfine state) are labelled by $\alpha = \uparrow\downarrow$ and assumed to have equal concentration. We choose a unit system in which $\hbar = \omega = m = 1$, where m is the atomic mass, and ω is the frequency of an isotropic trapping potential. We start from the Hamiltonian $H = H_0 + H_{int}$, where

$$H_0 = \sum_\alpha \int d\mathbf{r}\, \psi_\alpha^\dagger(\mathbf{r}) \left(-\frac{1}{2}\frac{d^2}{d\mathbf{r}^2} + \frac{1}{2}\mathbf{r}^2 \right) \psi_\alpha(\mathbf{r}) \tag{1}$$

is the one-particle part, $\psi_\alpha(\mathbf{r})$ being the field operators of the component α.

$$H_{int} = g \int d\mathbf{r}\, \psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}) \tag{2}$$

is the two-body interaction Hamiltonian, and g is the interaction strength. We do not use pseudopotential $\delta(\mathbf{r})\partial_r r$ [18–20], but use a delta function for simplicity. H_0 is easily diagonalized by writing the field operator as

$$\psi_\alpha(\mathbf{r}) = \prod_i^d \sum_{n^i} \frac{H_{n^i}(x_i)\,e^{-\frac{1}{2}x^2}}{\sqrt{\pi^{\frac{1}{2}} n^i! 2^{n^i}}} a_{\alpha n} \tag{3}$$

where d is the dimension of the system, $H_{n^i}(x_i)$ is the Hermite polynomial and $n = (n^1, \ldots)$ is quantum number, for example $n = (n^1, n^2, n^3)$ stand for $d = 3$, and $\left[a_{\alpha n}, a_{\alpha' n'}^\dagger\right] = \delta_{\alpha\alpha'}\delta_{nn'}$. Then H_0 and H_{int} are written in the form

$$H_0 = \sum_{i=1}^d \sum_{n^i \alpha} \left(n^i + \frac{1}{2} \right) a_{\alpha n}^\dagger a_{\alpha n} \tag{4}$$

$$H_{int} = g \prod_{i=1}^d \sum_{n_1^i n_2^i n_3^i n_4^i} w\left(n_1^i, n_2^i, n_3^i, n_4^i\right) a_{\uparrow n_1}^\dagger a_{\downarrow n_2}^\dagger a_{\downarrow n_3} a_{\uparrow n_4}. \tag{5}$$

Here, $w(n_1, n_2, n_3, n_4)$ is a function of quantum numbers of the trapped system and is given by

$$w(n_1, n_2, n_3, n_4) = (-1)^{\frac{-n_1+n_2-n_3+n_4}{2}} \sqrt{\frac{(2n_1)!!(2n_3)!!}{(2n_2)!!(2n_4)!!}}$$

$$\times \int_0^\infty \frac{dk}{\pi} k^{-n_1+n_2-n_3+n_4}\, e^{-\frac{k^2}{2}} L_{n_1}^{n_2-n_1}\left(\frac{k^2}{2}\right) L_{n_3}^{n_4-n_3}\left(\frac{k^2}{2}\right) \tag{6}$$

where $L_n^m(x)$ is the associated Laguerre polynomial. We can calculate this form using Fourier transformation [21, 22].

3. Ground-state properties

3.1. One-dimensional case

We use first-order perturbation theory, and obtain one-dimensional interaction energy for the one-dimensional case in the form

$$E_{\text{int}}(n_F) = g \sum_{n=0,m=0}^{n_F} w(n,m,m,n) \tag{7}$$

where n_F is defined by $\frac{\epsilon_F}{\hbar\omega} - d/2$, where ϵ_F is the Fermi energy. Henceforth, we call this Fermi number. The function $w(n,n,m,m)$ has a complicated dependence on quantum numbers, showing considerable difference from the momentum expansion ordinarily used to treat homogeneous systems. We can nevertheless calculate equation (7) exactly as shown in appendix A, with the result

$$E_{\text{int}}(n_F) = \frac{g}{\sqrt{2\pi}} \sum_{r=0}^{n_F} \left(\frac{(2r+1)!!}{(2r)!!} \right)^2 \frac{(2n_F - 2r - 1)!!}{(2n_F - 2r)!!}. \tag{8}$$

The above summation can be reduced further to an integral, yielding a simple form of energy

$$E(N) \cong \frac{N^2}{4} + \frac{4g}{3\pi^2} N^{\frac{3}{2}}. \tag{9}$$

Here, $N = 2N_\uparrow = 2N_\downarrow$ is the total number of particles. The chemical potential is also calculated by the relation $\mu(N_\alpha) = (E(N_\alpha + 1) - E(N_\alpha))/2$, yielding

$$\mu(N_\alpha) \cong N_\alpha + \frac{2g}{\pi^2} \sqrt{2N_\alpha}. \tag{10}$$

Equations (9), (10) show that as N increases, the ratio $E_{\text{int}}/(E - E_{\text{int}})$ becomes small, thus one may see that the system behaves like a free particle. However we should note the fact that, in general, even a weak interaction destroys the Fermi-liquid nature in one dimension. So, the above result may not be correct if we take quantum fluctuation into account. However, we do not intend to analyse specific features of one dimension. Our purpose in this study is to discuss arbitrary dimensions.

3.2. Two-dimensional case

In the two-dimensional case, we have to deal with the degeneracy which makes calculation of the energy rather complicated. However, as shown in appendix B, the total energy can be calculated exactly in this approximation, with the result

$$E(n_F) = \frac{1}{3}(n_F + 1)(n_F + 2)(2n_F + 3) + \frac{g}{12\pi}(n_F + 1)(n_F + 2)(2n_F + 3) \tag{11}$$

where we assume that the particle is filled up to the Fermi level. The Fermi number n_F and total number N are related by $n_F = \frac{1}{2}(-3 + \sqrt{4N + 1})$; therefore, the energy is given by

$$E(N) = \frac{N}{3} \sqrt{4N + 1} \left(1 + \frac{g}{4\pi} \right). \tag{12}$$

Both the single-particle term and the interaction term have the same dependence on the number of particles, similar to the results for the homogeneous system. The mean value of the chemical potential $\mu(N) = \frac{1}{2}\frac{\partial E}{\partial N}$ is given by

$$\mu(N_\alpha) = \frac{12N_\alpha + 1}{2\sqrt{8N_\alpha + 1}} \left(1 + \frac{g}{4\pi} \right). \tag{13}$$
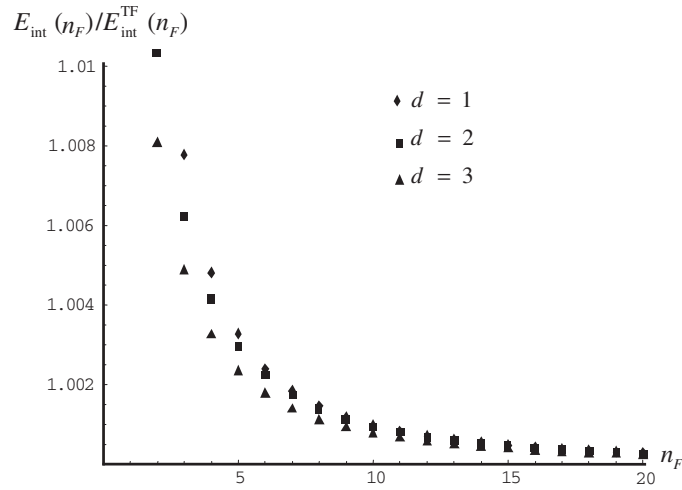
Figure 1. The ratio of the interaction energy to its Thomas–Fermi value is plotted as a function of Fermi number n_F. d represents the dimension of the system.

In the following, we investigate the one-particle energy shift. Let us add one more particle to the system filled up to the Fermi level. The energy shift $\xi(n_x, n_y)$ is generally a function of degenerate level n_x, n_y ($n_x + n_y = n_F + 1$). But in this case, it does not depend on the coordinate; in other words, interactions between particles belonging to different energy levels do not lift degeneracy [24]. This result is expected to have a connection to exclusion statistics in a two-dimensional gas [11–14]. $\xi(n_x, n_y)$ is given by

$$\xi(n_x, n_y) = \frac{g}{4\pi}(n_F + 1). \tag{14}$$

Note that the interacting term in equation (13) corresponds exactly to the above form.

3.3. Three-dimensional case

The calculation can be done more or less similarly also in the three-dimensional case. The details are given in appendix D. The result for total energy is given by

$$E(n_F) = \frac{1}{4}(n_F + 1)(n_F + 2)^2(n_F + 3) + \frac{g}{9(2\pi)^{\frac{3}{2}}} \sum_{r=0}^{n_F} \left(\frac{(2r+3)!!}{(2r)!!} \right)^2 \frac{(2n_F - 2r + 1)!!}{(2n_F - 2r)!!} \tag{15}$$

$$\cong \frac{3^{\frac{4}{3}}}{4} N^{\frac{4}{3}} + \frac{256\sqrt{6}g}{945\pi^3} N^{\frac{3}{2}}. \tag{16}$$

Equation (16) can also be obtained from the Thomas–Fermi approximation with weak interaction ($|g| \ll 1$) [15–17, 23]

$$E_{\text{int}}^{\text{TF}}(N) = \frac{g}{4} \int \mathrm{d}^3x \, n(x)^2 \tag{17}$$

where $n(x) = \frac{2^{3/2}}{3\pi^2}\left[\mu - \frac{1}{2}x^2\right]^{3/2}$ is the non-interacting particle density. It is possible to indicate the same procedure in the one- and two-dimensional case. As can be seen from figure 1, the exact first-order perturbation calculation for the interaction energies is very close to the Thomas–Fermi approximation. While the Thomas–Fermi approximation is ordinarily correct
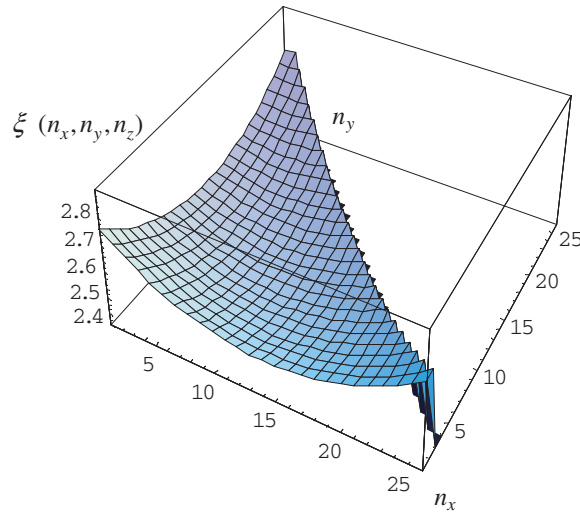
Figure 2. Single-particle energy shift of the three-dimensional system caused by inter-shell interaction for $g = 1$. $(n_x, n_y, n_z) = (n_x, n_y, n_F + 1 - n_x - n_y)$ is the degenerate state of the particle.

in a system of sufficiently large number of particles, our results show that this approximation is applicable even to a system which has a small number of particles. The average chemical potential is given by

$$\mu(N_\alpha) = (6N_\alpha)^{\frac{1}{3}}\left(1 + \frac{128\sqrt{2}g}{315\pi^3}(6N_\alpha)^{\frac{1}{6}}\right). \tag{18}$$

We investigate the energy shift more precisely as we have done in the two-dimensional cases. The one-particle energy shift $\xi(n_x, n_y, n_z)$ is found to be

$$\xi(n_x, n_y, n_z) = \frac{\sqrt{2}g}{4\pi^{\frac{3}{2}}} \frac{d^{n_F}}{d\beta^{n_F}}\left(\beta^{\frac{1}{2}+n_F}F_{n_x}(\beta)F_{n_y}(\beta)F_{n_z}(\beta)\right)\Big|_{\beta=1} \tag{19}$$

where $n_x, n_y, n_z(n_x + n_y + n_z = n_F + 1)$ are coordinates of degenerate states and $F_n(\beta)$ stands for hypergeometric function $F\left(-n, \frac{1}{2}, 1; \frac{1}{\beta}\right)$.

As shown in figure 2, the interaction lifts the degeneracy and equation (19) has maximum value when two of the quantum numbers are zero. The maximum value is

$$\xi(n_x, n_y, n_z) \cong \frac{\sqrt{2}g}{\pi^2}\left(\frac{4}{9\pi}n_F^{\frac{3}{2}} + \frac{65}{192}n_F^{\frac{1}{2}}\right). \tag{20}$$

This form reproduces the results of [15, 16]. Equation (19) has a minimum value at $n_x \cong n_y \cong n_z$, namely the one-particle wavefunction forms a symmetrical shape. We calculate the minimum value numerically and find it to be roughly 0.89 times lower than the maximum value.

4. Finite temperature

At finite temperatures, the number of particles and the energy of d-dimensional systems are written in the form

$$N = 2\sum_{n=0}^{\infty} D(n)f(n) \tag{21}$$

$$E = 2 \sum_{n=0}^{\infty} D(n) \left(n + \frac{d}{2} \right) f(n) + g \prod_{i=1}^{d} \sum_{\substack{n_i=0 \\ m_i=0}}^{\infty} h(n_i, m_i) f(n) f(m) \tag{22}$$

where $f(n)$ is the distribution function, $n = n_1 + \cdots + n_d$ and $D(n) = \sum_{n_1+\cdots+n_d=n} 1$ is the density of state. We restrict our discussion to high temperatures, i.e. well above the Fermi temperature. Then, we use the Boltzmann distribution function and calculate equation (21) and find it to be

$$\begin{aligned} N &= 2 \frac{e^{\frac{1}{T}(\mu - \frac{d}{2})}}{\left(1 - e^{-\frac{1}{T}}\right)^d} \\ &\cong 2 T^d \, e^{\frac{1}{T}(\mu - \frac{d}{2})}. \end{aligned} \tag{23}$$

Here, we put Boltzmann's constant $k_B = 1$ for simplicity. Using equation (21), we can calculate equation (22) which is given in appendix E with the result

$$\begin{aligned} E &= \frac{dN}{2 \tanh\left(\frac{2}{T}\right)} + \frac{g}{4(2\pi)^{\frac{d}{2}}} N^2 \tanh^{\frac{d}{2}}\left(\frac{2}{T}\right) \\ &\cong dNT + \frac{gN^2}{4(4\pi T)^{\frac{d}{2}}}. \end{aligned} \tag{24}$$

We have thus obtained both $T = 0$ and $T \to \infty$ limits. These results may be useful in constructing an interpolation formula for finite temperatures.

5. Summary

In summary, we have studied an interacting Fermi gas in an arbitrary dimensional isotropic harmonic trap. We have used first-order perturbation theory. In contrast to the homogeneous system, the first-order perturbation term has a non-trivial contribution to the system. Dependence on the number of particles, dimension and temperature is characteristic of the trapped system. We have calculated energy and chemical potential. As a result we found that the Thomas–Fermi approximation is applicable even if the system is small. Furthermore we found the interaction energy decreases in proportion to $T^{-\frac{d}{2}}$, when the temperature of the system is well above the Fermi temperature.

Appendix A. Derivation of equations (9), (10)

One easily finds that equation (7) is written in the form

$$E_{\text{int}}(n_F) = \frac{g}{\sqrt{2\pi}} \int_0^{\infty} dk \, k^{-\frac{1}{2}} \, e^{-k} \sum_{n=0}^{n_F} L_n(k) \sum_{m=0}^{n_F} L_m(k). \tag{A.1}$$

We use the relation

$$L_n^a(x) = \sum_{r=0}^{n} \frac{\Gamma(a - b + r)}{r! \, \Gamma(a - b)} L_{n-r}^b(x) \tag{A.2}$$

where $\Gamma(r)$ is the gamma function; we put $a = 1$ and $b = 0$ and find

$$E_{\text{int}}(n_F) = \frac{g}{\sqrt{2\pi}} \int_0^{\infty} dk \, k^{-\frac{1}{2}} \, e^{-k} L_{n_F}^1(k)^2. \tag{A.3}$$

We use equation (A.2) again, put $a = 1$ and $b = -\frac{1}{2}$ and use the relation

$$\int_0^\infty \mathrm{d}x \, \mathrm{e}^{-x} x^a L_n^a(x) L_m^a(x) = \frac{\Gamma(a+n+1)}{n!} \delta_{n,m}. \tag{A.4}$$

One easily finds equation (8) from equation (A.4). We use Wallis's formula $(2r-1)!!/(2r)!! \cong 1/\sqrt{\pi r}$ and replace the summation with an integral, then the interaction term of equation (8) is reduced to

$$\frac{4g}{\sqrt{2}\pi^2} \int_0^{n_F} \mathrm{d}r \, \frac{r}{\sqrt{n_F - r}}. \tag{A.5}$$

We obtain equation (9) directly from (A.5). The interaction term of the chemical potential is written in the form

$$\mu_{\mathrm{int}}(N_\alpha) = (E_{\mathrm{int}}(N_\alpha + 1) - E_{\mathrm{int}}(N_\alpha))/2$$

$$= g \sum_{n=0}^{n_F+1} w(n, n, n_F, n_F). \tag{A.6}$$

We use relations (A.2) and (A.4) and find

$$\mu_{\mathrm{int}}(N_\alpha) = \frac{g}{\sqrt{2\pi}} \sum_{r=0}^{n_F+1} \left(\frac{(2r-1)!!}{(2r)!!} \right)^2 \frac{(2n_F + 1 - 2r)!!}{(2n_F + 2 - 2r)!!} (2r+1)$$

$$\cong \frac{\sqrt{2}g}{\pi^2} \int_0^{n_F+1} \mathrm{d}r \, \frac{1}{\sqrt{n_F + 1 - r}}. \tag{A.7}$$

We obtain equation (10) directly from (A.7).

Appendix B. Derivation of two-dimensional energy

The two-dimensional interaction energy is written in the form

$$E_{\mathrm{int}}(N) = g \sum_{n=0}^{n_F} \sum_{m=0}^{n_F} \sum_{\substack{n_x+n_y=n \\ m_x+m_y=m}} h(n_x, m_x) h(n_y, m_y). \tag{B.1}$$

Here, $h(n, m)$ represents $w(n, n, m, m)$. We use the relation

$$\sum_{n_1+\cdots+n_d=n} L_{n_1}(x_1) \cdots L_{n_d}(x_d) = L_n^{d-1}(x_1 + \cdots + x_d). \tag{B.2}$$

We put $d = 2$, then equation (B.1) is reduced to

$$E_{\mathrm{int}}(N) = g \sum_{n=0}^{n_F} \sum_{m=0}^{n_F} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \mathrm{e}^{-\frac{1}{2}(x^2+y^2)} L_m^1 \left(\frac{x^2+y^2}{2} \right) L_n^1 \left(\frac{x^2+y^2}{2} \right). \tag{B.3}$$

We replace above the Cartesian coordinate values (x, y) with polar coordinates (r, θ) and use equation (A.2), then equation (B.3) is reduced to

$$E_{\mathrm{int}}(N) = \frac{g}{2\pi} \int_0^\infty \mathrm{d}\left(\frac{r^2}{2} \right) \mathrm{e}^{-\frac{r^2}{2}} \left(L_{n_F}^2 \left(\frac{r^2}{2} \right) \right)^2. \tag{B.4}$$

We apply equations (A.2) and (A.4) to the above form, and obtain equation (11). Note that three-dimensional energy can also be calculated.

Appendix C. Derivation of equations (18), (19)

Suppose that we are adding one more particle to the three-dimensional system whose particles are filled up to Fermi level. Then, the energy shift is

$$\xi(n_x, n_y, n_z) = \frac{g}{(2\pi)^3} \sum_{m=0}^{n_F} \sum_{m_x+m_y+m_z=m} h(m_x, n_x)h(m_y, n_y)h(m_z, n_z)$$

$$= \frac{g}{\pi^3} \int_0^\infty dr\, r^2 e^{-\frac{1}{2}r^2} \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \sin\theta L_{n_F}^3 \left(\frac{r^2}{2}\right)$$

$$\times L_{n_x}\left(\frac{r^2}{2}\sin^2\theta\cos^2\varphi\right) L_{n_y}\left(\frac{r^2}{2}\sin^2\theta\sin^2\varphi\right) L_{n_z}\left(\frac{r^2}{2}\cos^2\theta\right). \quad (C.1)$$

We expand the Laguerre polynomials and calculate angular integrals,

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \sin\theta L_{n_x}\left(\frac{r^2}{2}\sin^2\theta\cos^2\varphi\right) L_{n_y}\left(\frac{r^2}{2}\sin^2\theta\sin^2\varphi\right) L_{n_z}\left(\frac{r^2}{2}\cos^2\theta\right)$$

$$= \frac{1}{4} \sum_{l_1=0}^{n_x} \sum_{l_2=0}^{n_y} \sum_{l_3=0}^{n_z} (-1)^l \binom{n_x}{l_x}\binom{n_y}{l_y}\binom{n_z}{l_z} \frac{r^l}{l_x! l_y! l_z!} \frac{\Gamma(l_x+\frac{1}{2})\Gamma(l_y+\frac{1}{2})\Gamma(l_z+\frac{1}{2})}{\Gamma(l+\frac{3}{2})}$$

$$(C.2)$$

where $\binom{n}{l}$ is the binomial coefficient, and we put $l = l_x + l_y + l_z$. The radial integral is also calculated and found to be

$$\int_0^\infty dr\, r^{\frac{1}{2}+l} e^{-r} L_{n_F}^3(r) = \frac{\Gamma(\frac{5}{2}-l+n_F)}{n_F! \Gamma(\frac{5}{2}-l)} \Gamma\left(\frac{3}{2}+l\right). \quad (C.3)$$

From equations (C.2) and (C.3), we find

$$\xi(n_x, n_y, n_z) = \frac{g\sqrt{2}}{4\pi^3 n_F!} \sum_{l_1=0}^{n_1} b(n_x, l_x) \sum_{l_y=0}^{n_y} b(n_y, l_y) \sum_{l_z=0}^{n_z} b(n_z, l_z) \frac{\Gamma(\frac{5}{2}-l+n_F)}{\Gamma(\frac{5}{2}-l)} \quad (C.4)$$

where $b(n, l)$ is $(-1)^l \binom{n}{l}\Gamma(l+\frac{1}{2})/l!$. Equation (19) is derived from the above form. When $n_y = n_z = 0$, equation (C.4) is reduced to

$$\xi(n_F+1, 0, 0) = (n_F+1)\frac{\sqrt{2}g}{2\pi^{\frac{3}{2}}} \sum_{l=0}^{n_F+1} \frac{(2l-1)!!}{(2l)!!} \frac{(2l-5)!!}{(2l)!!} \frac{(2n_F-2l+3)!!}{(2n_F+2-2l)!!}. \quad (C.5)$$

We find that $\frac{(2l-1)!!}{(2l)!!}\frac{(2l-5)!!}{(2l)!!}$ decreases in proportion to l^{-3}, on the other hand $\frac{(2n_F-2l+3)!!}{(2n_F+2-2l)!!}$ decreases in propotion to $er(n_F-l)^{\frac{1}{2}}$ and has nearly constant value if $l \ll n_F$. Hence, we find

$$\xi(n_F+1, 0, 0) \cong (n_F+1)\frac{\sqrt{2}g}{2\pi^2} \sum_{l=0}^{n_F+1} \frac{(2l-1)!!}{(2l)!!} \frac{(2l-5)!!}{(2l)!!} \left(n_F^{\frac{1}{2}} + O\left(n_F^{-\frac{1}{2}}\right)\right)$$

$$\cong \frac{4\sqrt{2}}{9\pi^3} n_F^{\frac{3}{2}} + O\left(n_F^{\frac{1}{2}}\right). \quad (C.6)$$

Here, we use the relation $\pi \sum_{l=0}^\infty \frac{(2l-1)!!}{(2l)!!}\frac{(2l-5)!!}{(2l)!!} = 4/9$. We obtain the first term of equation (20). The coefficient of second term is approximated by summation of equation (C.5) up to $l = 2$.

Appendix D. Derivation of equation (24)

At high temperature, i.e. where the Boltzmann distribution is applicable, d-dimensional energy is given by

$$E_0 = 2 \sum_{n=0}^{\infty} D(n) \left(n + \frac{d}{2} \right) \tag{D.1}$$

$$= d \exp \left(\frac{1}{T} \left(\mu - \frac{d}{2} \right) \right) \frac{\left(1 + \exp \left(-\frac{1}{T} \right) \right)}{\left(1 - \exp \left(-\frac{1}{T} \right) \right)^{d+1}}$$

$$\cong d \exp \left(\frac{1}{T} \left(\mu - \frac{d}{2} \right) \right) \left(1 + \exp \left(-\frac{1}{T} \right) \right) T^{d+1}. \tag{D.2}$$

The kinetic term of equation (25) is obtained from equations (D.2) and (23). On the other hand the interaction term is given by

$$E_{\text{int}} = \frac{g}{(2\pi)^d} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\substack{n_1 + \cdots + n_d = n \\ m_1 + \cdots + m_d = m}} \exp \left(-\frac{1}{T} \left(n_1 + \cdots + n_d - \mu + \frac{d}{2} \right) \right)$$

$$\times \exp \left(-\frac{1}{T} \left(m_1 + \cdots + m_d - \mu + \frac{d}{2} \right) \right) \int_{-\infty}^{\infty} dx_1 \cdots dx_d$$

$$\times \exp \left(-\frac{1}{2} \left(x_1^2 + \cdots + x_d^2 \right) \right) L_{m_1} \left(\frac{x_1^2}{2} \right) \cdots L_{m_d} \left(\frac{x_d^2}{2} \right)$$

$$\times L_{n_1} \left(\frac{x_1^2}{2} \right) \cdots L_{n_d} \left(\frac{x_d^2}{2} \right). \tag{D.3}$$

We use relation (B.2) then the above form is reduced to

$$E_{\text{int}} = \frac{g}{(2\pi)^d} \frac{d\pi^{\frac{d}{2}}}{\Gamma \left(\frac{d}{2} + 1 \right)} \int_0^{\infty} dr \, r^{d-1} \exp \left(-\frac{r^2}{2} \right) \left(\sum_{n=0}^{\infty} \exp \left(-\frac{n}{T} \right) L_n^{d-1} \left(\frac{r^2}{2} \right) \right)^2. \tag{D.4}$$

We use equation (22) and the relation

$$\sum_{n=0}^{\infty} t^n L_n^{d-1}(x) = \frac{\exp \left(-\frac{xt}{1-t} \right)}{(1-t)^d} \tag{D.5}$$

and find

$$E_{\text{int}} = \frac{g}{(2\pi)^d} \frac{d\pi^{\frac{d}{2}}}{\Gamma \left(\frac{d}{2} + 1 \right)} N^2 \int_0^{\infty} dr \, r^{d-1} \exp \left(-\frac{r^2}{2 \tanh \left(\frac{1}{2T} \right)} \right). \tag{D.6}$$

We obtain equation (24) from equation (D.6).

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